

# Maximal sets of given diameter in the grid and the torus

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## Abstract

The grid graph is the graph on  $[k]^n = \{0, 1, \dots, k-1\}^n$  in which  $x = (x_1, \dots, x_n)$  is joined to  $y = (y_1, \dots, y_n)$  if for some  $j$  we have  $|x_j - y_j| = 1$  and  $x_i = y_i$  for all  $i \neq j$ . One of our aims in this paper is to determine, for each positive integer  $d$ , the maximum size of a subset of  $[k]^n$  of diameter  $d$ .

The discrete torus is the corresponding graph on  $\mathbb{Z}_k^n = (\mathbb{Z}/k\mathbb{Z})^n$ : a point  $x = (x_i)_i^n$  is joined to  $y = (y_i)_i^n$  if for some  $j$  we have  $x_j = y_j \pm 1$  and  $x_i = y_i$  for all  $i \neq j$ . Our other main aim is to determine, for each  $d$ , the maximum size of a subset of  $\mathbb{Z}_k^n$  of diameter  $d$ , in the case  $k$  even.

## 0. Introduction

Let  $[k] = \{0, 1, \dots, k-1\}$ , and define a graph on  $[k]^n$  by joining  $x = (x_1, \dots, x_n)$  to  $y = (y_1, \dots, y_n)$  if for some  $j$  we have  $|x_j - y_j| = 1$  and  $x_i = y_i$  for all  $i \neq j$ . We call this graph the *grid graph* or simply the *grid*  $[k]^n$ . Equivalently, the grid  $[k]^n$  is the product of  $n$  paths of order  $k$ . One of our aims in this paper is to determine the maximal size of a subset of  $[k]^n$  of given diameter. Recall that the diameter of a subset  $A$  of the vertex set of a graph  $G$  is  $\text{diam } A = \max\{d(x, y) : x, y \in A\}$ , where  $d(x, y)$  is the graph distance.

The question above was first considered by Kleitman and Fellows [7], although in the case  $k=2$  the problem had been completely solved by Kleitman [6] for  $d$  even, answering a question of Erdős. In order to state Kleitman and Fellows' result, let us define, for a point  $c = (c_1, \dots, c_n) \in [0, k-1]^n$  and  $r > 0$ , the *ball* of centre  $c$  and

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radius  $r$  to be

$$B(c, r) = B_{[k]^n}(c, r) = \left\{ x \in [k]^n : \sum |x_i - c_i| \leq r \right\}.$$

Note that the radius  $r$  and the coordinates  $c_i$  of the centre  $c$  are not assumed to be integers. With this notation, Kleitman and Fellows [7] proved that if  $d$  is even and at most  $n \lfloor k/2 \rfloor - 1$  then the ball  $B(c, d/2)$ , with  $c = (\lfloor (k-1)/2 \rfloor, \dots, \lfloor (k-1)/2 \rfloor)$ , has maximal size among all sets in  $[k]^n$  of diameter  $d$ . They also conjectured that for every natural number  $d$  there is a point  $c$ , whose coordinates need not be integers, such that the ball  $B(c, d/2)$  has maximal size among sets of diameter  $d$ . In Section 2 of this paper, we shall prove this conjecture, and determine an appropriate  $c$  in each case.

We remark that, in fact, Kleitman and Fellows proved their result for more general grids, namely products of paths that are not necessarily of the same length, and they made their conjecture about all such grids. However, in this paper we consider only grids  $[k]^n$  as above.

[*Note added during revisions:* We have recently been informed that some of our results in Section 2 have been obtained in two forthcoming papers: ‘Diameter and Radius in Manhattan Lattices’, by D.Z. Du and D.J. Kleitman, and ‘Diametric Theorems for Sequence Spaces’, by R. Ahlswede, C. Cai and Z. Zhang. To be precise, Du and Kleitman prove Theorems 8 and 10 (dealing with  $k$  even), and they also prove Theorem 11 ( $k$  odd) under the additional assumption that either  $d < n(k-1)/2$  or  $d \geq n(k-1)/2 + n - 1$ . Ahlswede, Cai and Zhang prove Theorem 11, and also prove Theorems 8 and 10 except in the case when  $d$  is an odd number with  $(n-1)k/2 \leq d < nk/2 - 1$ . The methods in these papers are rather different from the ones we use here.]

Our other main aim in this paper is to consider a problem that seems to be somewhat harder, namely the analogous question for the discrete torus  $\mathbb{Z}_k^n = (\mathbb{Z}/k\mathbb{Z})^n$ , which is the product of  $n$  cycles of order  $k$ . In Section 3 we give a complete solution to this problem for  $k$  even, and make some remarks about the case when  $k$  is odd.

Our attack on the problem for the torus is based on some isoperimetric inequalities. To prove these inequalities, we make use of fractional systems, which were introduced in [4]. The isoperimetric inequalities we obtain extend slightly those in [3]. As an easy application of fractional systems, we shall give a direct solution to the diameter problem in the infinite grid  $\mathbb{Z}^n$ , first proved by Kleitman and Fellows [7].

Our notation is fairly standard. The standard basis of  $\mathbb{Z}^n$  is written  $e_1, \dots, e_n$ ; for example, in  $[k]^n$  the vector  $(1, 0, 2, 0, \dots, 0)$  is denoted by  $e_1 + 2e_3$ . The complement of set  $I \subset \{1, \dots, n\}$  is written  $\bar{I}$ . We often suppress brackets, writing  $\bar{i}$  for  $\{\bar{i}\}$ . Thus for example  $\mathbb{Z}^{\bar{i}}$  denotes  $\{x \in \mathbb{Z}^n : x_i = 0\}$ .

Given real-valued functions  $f, g$  on a set  $X$ , we write  $f \leq g$  if  $f(x) \leq g(x)$  for all  $x \in X$ . The function  $f \vee g$  is defined by  $(f \vee g)(x) = \max(f(x), g(x))$ . More generally, given functions  $f_i : X \rightarrow \mathbb{R}$  for  $i \in I$ , we write  $\bigvee \{f_i : i \in I\}$  for the function  $f$  given by  $f(x) = \sup \{f_i(x) : i \in I\}$ . We say that the  $f_i$  are *nested* if for any  $i$  and  $j$  we have  $f_i \leq f_j$  or  $f_j \leq f_i$ .

The *support* of a function  $f: X \rightarrow \mathbb{R}$  is  $\text{supp } f = \{x \in X: f(x) \neq 0\}$ . If  $f$  has finite support, we define the *weight* of  $f$  as  $w(f) = \sum_{x \in \text{supp } f} f(x)$ .

## 1. Subsets of $\mathbb{Z}^n$

We shall start by introducing fractional systems and using them to give a direct proof of the result of Kleitman and Fellows [7] concerning the maximal size of sets of given diameter in the infinite grid  $\mathbb{Z}^n$ .

A *fractional system*, or simply a *system*, on  $\mathbb{Z}^n$  is a function from  $\mathbb{Z}^n$  to the closed interval  $[0, 1]$ . Note that a fractional system on  $\mathbb{Z}^n$  is a generalization of a subset of  $\mathbb{Z}^n$ : if a function  $f: \mathbb{Z}^n \rightarrow [0, 1]$  is such that  $f(\mathbb{Z}^n) \subset \{0, 1\}$  then  $f$  is naturally identified with the subset  $A = f^{-1}(1)$  of  $\mathbb{Z}^n$ . Fractional systems first appeared in [4].

We shall only be concerned with systems whose support is finite — for brevity we omit this condition in the sequel. The *diameter* of a system  $f$  is

$$\text{diam } f = \max \{d(x, y) + f(x) + f(y) - 2: x, y \in \text{supp } f\},$$

with the convention that if  $f \equiv 0$  then  $\text{diam } f = -\infty$ . Thus if  $f(\mathbb{Z}^n) \subset \{0, 1\}$  and  $f$  is identified with  $A = f^{-1}(1)$  then  $\text{diam } f$  is just the usual diameter of  $A$ .

We remark here that there are other plausible definitions of the diameter for fractional systems. For example, one could take  $\text{diam } f = \max \{d(x, y) f(x) f(y): x, y \in \text{supp } f\}$ . Such alternatives could perhaps be useful in other situations.

A system  $f$  which is of the form

$$f(x) = \begin{cases} 1 & \text{if } d(x, 0) < r, \\ \alpha & \text{if } d(x, 0) = r, \\ 0 & \text{if } d(x, 0) > r, \end{cases}$$

for some  $r \in \mathbb{Z}_+$  and  $\alpha \in [0, 1]$  is called a *fractional ball*. For  $v \geq 0$  we write  $b^v$  for the (unique) fractional ball of weight  $v$ . Our aim is to show that, among systems of given diameter, a fractional ball has the greatest weight.

Let  $f$  be a system on  $\mathbb{Z}^n$ . For  $1 \leq i \leq n$  and  $x \in \mathbb{Z}^i$ , the *i-section* of  $f$  at  $x$  is the system  $f_{i|x}$  on  $\mathbb{Z}$  given by

$$f_{i|x}(y) = f(x + ye_i), \quad y \in \mathbb{Z}.$$

We wish to define a compression operator  $C_i$  which acts on a system  $f$  by ‘compressing’ each *i-section* of  $f$  into a (1-dimensional) fractional ball of the same weight. Thus, for a system  $f$  on  $\mathbb{Z}^n$ , we define the system  $C_i(f)$ , the *i-compression* of  $f$ , by giving its *i-sections*:

$$C_i(f)_{i|x} = b^{w(f_{i|x})}, \quad x \in \mathbb{Z}^i.$$

Thus  $w(C_i(f)) = w(f)$ . We say that  $f$  is *i-compressed* if  $C_i(f) = f$ .

We have the following easy lemma; although the proof given looks cumbersome, it is extremely simple.

**Lemma 1.** *Let  $f$  be a system on  $\mathbb{Z}^n$ , and let  $1 \leq i \leq n$ . Then  $\text{diam } C_i(f) \leq \text{diam } f$ .*

**Proof.** For convenience, write  $g$  for  $C_i(f)$ . It is sufficient to show that for each  $x, y \in \mathbb{Z}^i$  with  $w(f_{i|x}), w(f_{i|y}) > 0$  we have

$$\begin{aligned} & \max \{ |a-b| + f_{i|x}(a) + f_{i|y}(b) : a \in \text{supp } f_{i|x}, b \in \text{supp } f_{i|y} \} \\ & \geq \max \{ |a-b| + g_{i|x}(a) + g_{i|y}(b) : a \in \text{supp } g_{i|x}, b \in \text{supp } g_{i|y} \}. \end{aligned} \quad (1)$$

Fix then arbitrary points  $x, y \in \mathbb{Z}^i$  with  $w(f_{i|x}), w(f_{i|y}) > 0$ : say  $w(f_{i|x}) = r, w(f_{i|y}) = s$ , where without loss of generality we may assume  $r \leq s$ .

Put

$$\begin{aligned} p &= \min \text{supp } f_{i|x}, & p' &= \max \text{supp } f_{i|x}, \\ q &= \min \text{supp } f_{i|y}, & q' &= \max \text{supp } f_{i|y}, \end{aligned}$$

so that

$$\begin{aligned} & \max \{ |a-b| + f_{i|x}(a) + f_{i|y}(b) : a \in \text{supp } f_{i|x}, b \in \text{supp } f_{i|y} \} \\ & = \max(|p'-q| + f_{i|x}(p') + f_{i|y}(q), |q'-p| + f_{i|x}(p) + f_{i|y}(q')). \end{aligned} \quad (2)$$

Now, if  $p < p'$  then  $r \leq p' - p + f_{i|x}(p) + f_{i|y}(p') - 1$ , while if  $p = p'$  then  $r = f_{i|x}(p) = \frac{1}{2}(p' - p + f_{i|x}(p) + f_{i|x}(p'))$ . Since  $r > 1$  implies  $p < p'$ , it follows that

$$p' - p + f_{i|x}(p) + f_{i|x}(p') \geq \begin{cases} r+1 & \text{if } r > 1, \\ 2r & \text{if } r \leq 1. \end{cases}$$

Similarly, we have

$$q' - q + f_{i|y}(q) + f_{i|y}(q') \geq \begin{cases} s+1 & \text{if } s > 1, \\ 2s & \text{if } s \leq 1, \end{cases}$$

and so

$$\begin{aligned} & p' - p + q' - q + f_{i|x}(p) + f_{i|x}(p') + f_{i|y}(q) + f_{i|y}(q') \\ & \geq \begin{cases} r+1+s+1 & \text{if } r, s > 1, \\ 2r+s+1 & \text{if } r \leq 1, s > 1, \\ 2r+2s & \text{if } r, s \leq 1. \end{cases} \end{aligned}$$

It follows from (2) that

$$\begin{aligned} & \max \{ |a-b| + f_{i|x}(a) + f_{i|y}(b) : a \in \text{supp } f_{i|x}, b \in \text{supp } f_{i|y} \} \\ & \geq \begin{cases} \frac{r+1}{2} + \frac{s+1}{2} & \text{if } r, s > 1, \\ r + \frac{s+1}{2} & \text{if } r \leq 1, s > 1, \\ r+s & \text{if } r, s \leq 1. \end{cases} \end{aligned}$$

A similar relation holds with  $g$  in place of  $f$ , with the inequality replaced by equality. This establishes (1), as required.  $\square$

We can now prove that, among systems of given diameter, a fractional ball has the greatest weight.

**Theorem 2.** *Let  $f$  be a system on  $\mathbb{Z}^n$  of diameter  $d$ . Then  $w(f) \leq w(b)$ , where  $b$  is the fractional ball of diameter  $d$ .*

**Proof.** Let  $g = C_n(C_{n-1}(\dots C_1(f)\dots))$ . Then  $w(g) = w(f)$  and  $\text{diam } g \leq d$ . Now,  $g$  is  $i$ -compressed for all  $i$ , from which we have  $g(x) = g(-x)$  for all  $x \in \mathbb{Z}^n$ . Thus  $g(x) > 0$  implies  $2d(x, 0) + 2g(x) - 2 \leq \text{diam } g$ . It follows that  $g \leq b$ , and so  $w(f) = w(g) \leq w(b)$ , as required.  $\square$

In particular, we have the following result for subsets of  $\mathbb{Z}^n$ , first proved by Kleitman and Fellows [7].

**Corollary 3.** *Let  $A$  be a subset of  $\mathbb{Z}^n$  of diameter  $d$ , where  $d$  is even. Then  $|A| \leq |B(0, d/2)|$ .*

We now turn to the case of odd diameter. A system  $f$  on the line  $\mathbb{Z}$  which is of the form

$$f(x) = \begin{cases} 1 & \text{if } d(x, 1/2) < r + 1/2, \\ \alpha & \text{if } d(x, 1/2) = r + 1/2, \\ 0 & \text{if } d(x, 1/2) > r + 1/2, \end{cases}$$

for some  $r \in \mathbb{Z}_+$  and  $\alpha \in [0, 1]$  is called a *shifted fractional ball*. For  $v \geq 0$ , write  $s^v$  for the (unique) shifted fractional ball of weight  $v$ .

We wish to define a compression operator  $D_i$  which acts on a system  $f$  by replacing each  $i$ -section of  $f$  with a shifted fractional ball of the same weight. Thus, for a system  $f$  on  $\mathbb{Z}^n$ , the system  $D_i(f)$ , the *shifted  $i$ -compression* of  $f$ , is defined by:

$$D_i(f)_{i|x} = s^{w(f_{i|x})}, \quad x \in \mathbb{Z}^i.$$

Thus  $w(D_i(f)) = w(f)$ .

It is certainly not the case that  $\text{diam } D_i(f) \leq \text{diam } f$  for all systems  $f$ . Fortunately, however, the following lemma holds.

**Lemma 4.** *Let  $f$  be a system on  $\mathbb{Z}^n$  with  $f(\mathbb{Z}^n) \subset \{0, 1\}$ , and let  $1 \leq i \leq n$ . Then  $\text{diam } D_i(f) \leq \text{diam } f$ .*

**Proof.** Write  $g$  for  $D_i(f)$ . It is sufficient to show that for each  $x, y \in \mathbb{Z}^i$  with  $w(f_{i|x}), w(f_{i|y}) > 0$  we have

$$\begin{aligned} & \max \{ |a - b| + f_{i|x}(a) + f_{i|y}(b) : a \in \text{supp } f_{i|x}, b \in \text{supp } f_{i|y} \} \\ & \geq \max \{ |a - b| + g_{i|x}(a) + g_{i|y}(b) : a \in \text{supp } g_{i|x}, b \in \text{supp } g_{i|y} \}. \end{aligned}$$

Fix then an arbitrary  $x, y \in \mathbb{Z}^i$  with  $w(f_{i|x}), w(f_{i|y}) > 0$ : say  $w(f_{i|x}) = r$ ,  $w(f_{i|y}) = s$ . Then  $r, s \geq 1$  and so, exactly as in the proof of Lemma 1, we have

$$\max \{ |a - b| + f_{i|x}(a) + f_{i|y}(b) : a \in \text{supp } f_{i|x}, b \in \text{supp } f_{i|y} \} \geq \frac{r+1}{2} + \frac{s+1}{2}.$$

However, since  $r, s \in \mathbb{Z}$ , it is easy to see that a similar relation holds with  $g$  in place of  $f$  and with the inequality replaced by equality.  $\square$

We are now ready to find the maximal size of a set of given odd diameter in  $\mathbb{Z}^n$ .

**Theorem 5.** *Let  $A$  be a subset of  $\mathbb{Z}^n$  of diameter  $d$ , where  $d$  is odd. Then  $|A| \leq |B(c, d/2)|$ , where  $c = \frac{1}{2} e_1 = (\frac{1}{2}, 0, \dots, 0)$ .*

**Proof.** Let  $g = C_n(C_{n-1}(\dots C_2(D_1(A)) \dots))$ . Then  $w(g) = |A|$  and  $\text{diam } g \leq d$ . We have  $g(x) = g(e_1 - x)$  for all  $x \in \mathbb{Z}^n$ , so that  $g(x) > 0$  implies  $2d(x, c) + 2g(x) - 2 \leq \text{diam } g$ . It follows that  $g \leq B(c, d/2)$ , as required.  $\square$

## 2. Subsets of $[k]^n$

We now turn our attention to the finite grid  $[k]^n$ . Although we shall not be concentrating on fractional systems in this section, we start by showing how they give a direct solution to the diameter problem for the case of even diameter in the odd grid.

**Theorem 6.** *Let  $k$  be odd, and let  $A$  be a subset of  $[k]^n$  of diameter  $d$ , where  $d$  is even. Then  $|A| \leq |B(c, d/2)|$ , where  $c = ((k-1)/2, \dots, (k-1)/2)$ .*

**Proof.** Note that if  $f$  is a system on  $\mathbb{Z}^n$  with

$$\text{supp } f \subset \left\{ -\frac{k-1}{2}, \dots, \frac{k-1}{2} \right\}^n$$

then for any  $1 \leq i \leq n$  we have also

$$\text{supp } C_i(f) \subset \left\{ -\frac{k-1}{2}, \dots, \frac{k-1}{2} \right\}^n.$$

The result now follows exactly as in the proof of Theorem 2.  $\square$

In fact, the above method proves rather more. A subset  $S$  of  $\mathbb{Z}^n$  is called *unconditional* if whenever  $x \in S$  and  $|y_i| \leq |x_i|$  for all  $i$  then also  $y \in S$ .

**Theorem 7.** *Let  $S$  be an unconditional subset of  $\mathbb{Z}^n$ , and let  $A$  be a subset of  $S$  of diameter  $d$ , where  $d$  is even. Then*

$$|A| \leq \left| \left\{ x \in S : \sum |x_i| \leq d/2 \right\} \right|.$$

Our main method in this section will be the comparison of ‘opposite quadrants’ of the grid. We start with the even grid.

Let  $k$  be even, and let  $c = ((k-1)/2, \dots, (k-1)/2)$ . Thus  $c$  is the geometric centre of  $[k]^n$ . For any set  $A \subset [k]^n$ , and  $\varepsilon = (\varepsilon_i)_i \in \{-1, 1\}^n$ , we define the  $\varepsilon$ -quadrant of  $A$  to be

$$A_\varepsilon = \left\{ x \in \left[ \frac{k}{2} \right]^n : c + \sum \varepsilon_i (x_i + \frac{1}{2}) e_i \in A \right\}.$$

Note that all quadrants of  $A \subset [k]^n$  are subsets of  $[k/2]^n$ , so in particular  $A$  is not (in general) the union of its quadrants; however  $|A| = \sum_\varepsilon |A_\varepsilon|$ .

For  $0 \leq r \leq n(k/2 - 1)$ , let

$$F(r) = \left| \left\{ z \in \left[ \frac{k}{2} \right]^n : \sum z_i = r \right\} \right|.$$

Thus  $F(r) = F(n(k/2 - 1) - r)$  for all  $r$ . It is easy to check that the function  $F$  is increasing for  $0 \leq r \leq n(k/2 - 1)/2$ , for example, because  $[k/2]^n$  has a symmetric chain decomposition (see [1, Ch. 3]).

We are now ready to consider the simplest case for our method, namely the case of large diameter in the even grid.

**Theorem 8.** *Let  $k$  be even, and let  $A$  be a subset of  $[k]^n$  of diameter  $d$ , where  $d \geq nk/2$ . Then*

$$|A| \leq \begin{cases} |B(c, d/2)| & \text{if } d \equiv n \pmod{2} \\ |B(c + \frac{1}{2}e_1, d/2)| & \text{if } d \not\equiv n \pmod{2}, \end{cases}$$

where  $c = ((k-1)/2, \dots, (k-1)/2)$ .

**Proof.** Let us start with the case when  $d \equiv n \pmod{2}$ . For convenience, write  $B$  for  $B(c, d/2)$ . In order to show that  $|A| \leq |B|$ , we shall show that  $|A_\varepsilon| + |A_{-\varepsilon}| \leq |B_\varepsilon| + |B_{-\varepsilon}|$  for all  $\varepsilon$ .

Fix then an arbitrary  $\varepsilon \in \{-1, 1\}^n$ . If either of  $A_\varepsilon, A_{-\varepsilon}$  is empty then certainly  $|A_\varepsilon| + |A_{-\varepsilon}| \leq |B_\varepsilon| + |B_{-\varepsilon}|$ , since the fact that  $d \geq nk/2$  implies that  $|B_\varepsilon| + |B_{-\varepsilon}| \geq (k/2)^n$ . Hence we may assume that  $A_\varepsilon, A_{-\varepsilon} \neq \emptyset$ . Choose  $x \in A_\varepsilon$  and  $y \in A_{-\varepsilon}$  with  $\sum x_i$  and  $\sum y_i$  maximal: say  $\sum x_i = s$ ,  $\sum y_i = t$ . Since  $\text{diam } A = d$ , we must have  $s + t + n \leq d$ .

Now, by the choice of  $s$  and  $t$  we have

$$|A_\varepsilon| + |A_{-\varepsilon}| \leq \sum_0^s F(r) + \sum_0^t F(r).$$

Since  $s+t \leq d-n$ , and  $d-n \geq n(k/2-1)$ , we have

$$\sum_0^s F(r) + \sum_0^t F(r) \leq 2 \sum_0^{(d-n)/2} F(r) = |B_\varepsilon| + |B_{-\varepsilon}|.$$

Hence  $|A_\varepsilon| + |A_{-\varepsilon}| \leq |B_\varepsilon| + |B_{-\varepsilon}|$ , and so  $|A| \leq |B|$ .

The case  $d \not\equiv n \pmod{2}$  is very similar. Writing  $B$  for  $B(c + \frac{1}{2}e_1, d/2)$ , it is sufficient to show that  $|A_\varepsilon| + |A_{-\varepsilon}| \leq |B_\varepsilon| + |B_{-\varepsilon}|$  for all  $\varepsilon$ .

Fix then an arbitrary  $\varepsilon \in \{-1, 1\}^n$ . Since  $d \geq nk/2$ , it follows that  $|B_\varepsilon| + |B_{-\varepsilon}| \geq (k/2)^n$ , so we may again assume that  $A_\varepsilon, A_{-\varepsilon} \neq \emptyset$ . Choosing  $s$  and  $t$  as above, we have

$$|A_\varepsilon| + |A_{-\varepsilon}| \leq \sum_0^s F(r) + \sum_0^t F(r).$$

However, since  $s+t \leq d-n$ , and  $d-n \geq n(k/2-1)$ , we have

$$\sum_0^s F(r) + \sum_0^t F(r) \leq \sum_0^{(d-n+1)/2} F(r) + \sum_0^{(d-n-1)/2} F(r) = |B_\varepsilon| + |B_{-\varepsilon}|,$$

so that  $|A_\varepsilon| + |A_{-\varepsilon}| \leq |B_\varepsilon| + |B_{-\varepsilon}|$ , as required.  $\square$

To deal with smaller diameters in the even grid, and also for the odd grid, we shall need to compress our subsets towards the centre of the grid.

For any  $k$ , define an ordering  $<$  on  $[k]$  by setting  $x < y$  if either

$$\left| \frac{k-1}{2} - x \right| < \left| \frac{k-1}{2} - y \right|$$

or

$$\left| \frac{k-1}{2} - x \right| = \left| \frac{k-1}{2} - y \right| \quad \text{and} \quad x < \frac{k-1}{2} < y.$$

Thus for example if  $k$  is even then we have

$$\frac{k}{2} - 1 < \frac{k}{2} - 2 < \dots < k - 2 < 0 < k - 1.$$

We write  $I(a) = I_k(a)$  for the set of the first  $a$  elements in the  $<$  order on  $[k]$ .

Let  $A$  be a subset of  $[k]^n$ . For  $1 \leq i \leq n$  and  $x \in [k]^i$ , the  $i$ -section of  $A$  at  $x$  is the set

$$A_{i|x} = \{y \in [k]: x + ye_i \in A\}.$$

We wish to define a compression operator  $R_i$  which acts on a set  $A \subset [k]^n$  by compressing each  $i$ -section of  $A$  into a set of the same size of the form  $I(a)$ . More precisely, for any set  $A \subset [k]^n$ , and  $1 \leq i \leq n$ , we define the set  $R_i(A) \subset [k]^n$  by giving its  $i$ -sections:

$$R_i(A)_{i|x} = I(|A_{i|x}|), \quad x \in [k]^i.$$



Thus  $R_i$  is just a non-fractional version of the operator  $C_i$  of the previous section. Note that  $|R_i(A)| = |A|$ . We say that  $A$  is *compressed* if  $R_i(A) = A$  for all  $i$ .

We have the following very simple lemma.

**Lemma 9.** (i) Let  $A$  be a subset of  $[k]^n$ , and let  $1 \leq i \leq n$ . Then  $\text{diam } R_i(A) \leq \text{diam } A$ .  
(ii) Let  $A$  be a subset of  $[k]^n$ . Then there is a compressed set  $A' \subset [k]^n$  which satisfies  $|A'| = |A|$  and  $\text{diam } A' \leq \text{diam } A$ .

**Proof.** (i) Mirror the proof of Lemma 1.

(ii) The set  $A' = R_n(R_{n-1}(\dots R_1(A) \dots))$  satisfies  $|A'| = |A|$  and  $R_i(A') = A'$  for all  $i$ . By part (i), we have  $\text{diam } A' \leq \text{diam } A$ .  $\square$

We need a small amount of further notation. Let  $k$  be even, and let  $A$  be a subset of  $[k]^n$ . For  $\varepsilon \in \{-1, 1\}^n$ , define

$$s(\varepsilon) = s_A(\varepsilon) = \begin{cases} \max \left\{ \sum x_i : x \in A_\varepsilon \right\} & \text{if } A_\varepsilon \neq \emptyset, \\ -1 & \text{if } A_\varepsilon = \emptyset. \end{cases}$$

Thus

$$|A| \leq \sum_{\varepsilon} \sum_{r=0}^{s(\varepsilon)} F(r). \quad (3)$$

Note that if  $A$  is compressed then the function  $s(\varepsilon)$  does not vary too sharply. Indeed, let us regard  $\{-1, 1\}^n$  as a copy of the discrete cube (joining  $\varepsilon$  to  $\varepsilon'$  if for some  $i$  we have  $\varepsilon_i = -\varepsilon'_i$  and  $\varepsilon_j = \varepsilon'_j$  for all  $j \neq i$ ). Then it is easy to see that if  $A$  is a compressed set and  $\varepsilon, \varepsilon' \in \{-1, 1\}^n$  are adjacent then we have

$$|s_A(\varepsilon') - s_A(\varepsilon)| \leq 1. \quad (4)$$

For any integer  $a$ , we write  $A^{(\geq a)}$  for  $\{\varepsilon \in \{-1, 1\}^n : s_A(\varepsilon) \geq a\}$ .

We are now ready to deal with the case of small diameter in the even grid.

**Theorem 10.** Let  $k$  be even, and let  $A$  be a subset of  $[k]^n$  of diameter  $d$ , where  $d < nk/2$ . Then

$$|A| \leq \begin{cases} |B(c', d/2)| & \text{if } d \text{ is even,} \\ |B(c' + \frac{1}{2}e_1, d/2)| & \text{if } d \text{ is odd,} \end{cases}$$

where  $c' = (k/2, \dots, k/2)$ .

**Proof.** By Lemma 9, we may assume without loss of generality that  $A$  is compressed. For convenience, set  $B = B(c', d/2)$  if  $d$  is even and  $B = B(c' + \frac{1}{2}e_1, d/2)$  if  $d$  is odd.

We distinguish two cases, according to whether  $d \geq 2n$  or  $d < 2n$ . The latter case will be almost the same as the former, but with a slight extra complication.

Case (i):  $d \geq 2n$ .

Since  $\text{diam } A \leq d$ , we have  $s(\varepsilon) + s(-\varepsilon) = s_A(\varepsilon) + s_A(-\varepsilon) \leq d - n$  whenever  $s(\varepsilon), s(-\varepsilon) \neq -1$ . Also, if  $s(\varepsilon) > d/2$  for some  $\varepsilon$  then from (4) we have  $s(-\varepsilon) > d/2 - n \geq 0$ , which contradicts  $s(\varepsilon) + s(-\varepsilon) \leq d - n$ .

So  $s(\varepsilon) \leq d/2$  for all  $\varepsilon$ , and also  $s(\varepsilon) + s(-\varepsilon) \leq d - n$  for all  $\varepsilon$ . Hence, rearranging the sum in (3), we have

$$|A| \leq \begin{cases} \alpha_1 + \sum_{a=(d-n)/2+1}^{\lfloor d/2 \rfloor} |A^{(\geq a)}| (F(a) - F(d-n+1-a)) & \text{if } d \equiv n \pmod{2} \\ \alpha_2 + \sum_{a=(d-n+1)/2+1}^{\lfloor d/2 \rfloor} |A^{(\geq a)}| (F(a) - F(d-n+1-a)) & \text{if } d \not\equiv n \pmod{2}, \end{cases}$$

where

$$\alpha_1 = 2^n \sum_0^{(d-n)/2} F(r)$$

and

$$\alpha_2 = 2^n \sum_0^{(d-n-1)/2} F(r) + 2^{n-1} F((d-n+1)/2).$$

A similar relation holds with  $A$  replaced by  $B$ , with the inequality being replaced by equality. Now, since  $d < nk/2$ , for any  $a \geq (d-n)/2 + 1$  we have  $F(a) - F(d-n+1-a) \geq 0$ . Hence, to complete the proof, it will suffice to show that  $|A^{(\geq a)}| \leq |B^{(\geq a)}|$  for all integers  $a$  with  $(d-n)/2 + 1 \leq a \leq d/2$ .

Let then  $a$  be an integer with  $(d-n)/2 + 1 \leq a \leq d/2$ . Since  $s(\varepsilon) \geq a$  implies  $s(-\varepsilon) \leq d - n - a$ , equation (4) tells us that the sets  $A^{(\geq a)}$  and  $-A^{(\geq a)}$  are a distance of at least  $2a + n - d$  apart in  $\{-1, 1\}^n$ .

Now, if  $d$  is even, then Harper's vertex-isoperimetric inequality in the cube [5] (see [2, Ch. 16] for a general background) implies that if two sets in the discrete cube  $\{-1, 1\}^n$  are at least  $2a + n - d$  apart then they cannot both have size greater than  $\sum_{r=0}^{d/2-a} \binom{n}{r}$ , and so  $|A^{(\geq a)}| \leq |B^{(\geq a)}|$ . Similarly, if  $d$  is odd then Harper's theorem implies that if two sets in  $\{-1, 1\}^n$  are at least  $2a + n - d$  apart then they cannot both have size greater than

$$\sum_{r=0}^{(d-1)/2-a} \binom{n}{r} + \binom{n-1}{(d-1)/2-a},$$

and so again  $|A^{(\geq a)}| \leq |B^{(\geq a)}|$ .

Case (ii):  $d < 2n$ .

If  $s(\varepsilon), s(-\varepsilon) \geq 0$  then  $s(\varepsilon) + s(-\varepsilon) \leq d - n$ . Hence, rearranging the sum in (3), we have

$$|A| \leq \alpha_1 + \sum_{a=(d-n)/2+1}^{d-n} |A^{(\geq a)}| (F(a) - F(d-n+1-a)) + \sum_{\substack{a > d-n \\ a \geq 0}} |A^{(\geq a)}| F(a)$$

if  $d \equiv n \pmod{2}$ , and

$$|A| \leq \alpha_2 + \sum_{a=(d-n+1)/2+1}^{d-n} |A^{(\geq a)}| (F(a) - F(d-n+1-a)) + \sum_{\substack{a > d-n \\ a \geq 0}} |A^{(\geq a)}| F(a)$$

if  $d \not\equiv n \pmod{2}$ , where  $\alpha_1$  and  $\alpha_2$  are as in Case (i). Note that of course the first sum in each expression is 0 if  $d < n$ .

Now, a similar relation holds with  $B$  in place of  $A$ , with the inequality replaced by equality. Hence, to show that  $|A| \leq |B|$ , it suffices to show that  $|A^{(\geq a)}| \leq |B^{(\geq a)}|$  for all nonnegative  $a$  with  $a \geq (d-n)/2 + 1$ .

Let then  $a$  be a non-negative integer with  $a \geq (d-n)/2 + 1$ . If  $a \leq d/2$  then we obtain  $|A^{(\geq a)}| \leq |B^{(\geq a)}|$  exactly as in Case (i). So we may assume that  $a > d/2$ .

We claim that the distance in  $\{-1, 1\}^n$  between  $A^{(\geq a)}$  and  $-A^{(\geq a)}$  is at least  $n - d + 2a$ . Indeed, if this is not the case then there are  $\varepsilon, \varepsilon' \in \{-1, 1\}^n$  with  $s(\varepsilon), s(-\varepsilon) \geq a$  and  $d(\varepsilon, \varepsilon') > d - 2a$ . However, by (4) this implies that if  $d < n$  then there are  $\varepsilon'', \varepsilon'''$  with  $d(\varepsilon'', \varepsilon''') > d$  and  $s(\varepsilon''), s(\varepsilon''') \geq 0$ , while if  $d \geq n$  then there are  $\varepsilon'', \varepsilon'''$  with  $\varepsilon'' = -\varepsilon'''$  and  $s(\varepsilon'') > (d-n)/2$  and  $s(\varepsilon''') \geq (d-n)/2$ . In each case we contradict  $\text{diam } A \leq d$ .

So the distance between  $A^{(\geq a)}$  and  $-A^{(\geq a)}$  is at least  $n - d + 2a$ . Just as in Case (i), Harper's theorem now implies that if  $d$  is even then

$$|A^{(\geq a)}| \leq \sum_{r=0}^{d/2-a} \binom{n}{r},$$

while if  $d$  is odd then

$$|A^{(\geq a)}| \leq \sum_{r=0}^{(d-1)/2-a} \binom{n}{r} + \binom{n-1}{(d-1)/2-a}.$$

Thus  $|A^{(\geq a)}| \leq |B^{(\geq a)}|$ , as required.  $\square$

We now turn our attention to the odd grid. We first introduce some notation which is similar to that used for the even grid.

Let  $k$  be odd, and let  $c' = (k/2 - 1, \dots, k/2 - 1)$ . Our quadrants in  $[k]^n$  will be centred at  $c'$ , rather than at the geometric centre of  $[k]^n$ . For  $\varepsilon \in \{-1, 1\}^n$ , set

$$Q_\varepsilon = \prod_{i=1}^n [(k + \varepsilon_i)/2] = \{x \in \mathbb{Z}^n : 0 \leq x_i \leq (k + \varepsilon_i)/2 \text{ for all } i\},$$

and for any integer  $r$  let

$$F_\varepsilon(r) = |\{x \in Q_\varepsilon : \sum x_i = r\}|.$$

Writing  $|\varepsilon|$  for  $|\{i : \varepsilon_i = 1\}|$ , it is clear that  $F_\varepsilon$  depends only on  $|\varepsilon|$ . Note that if  $|\varepsilon| \leq |\varepsilon'|$  then for any  $r$  we have  $F_\varepsilon(r) \leq F_{\varepsilon'}(r)$  and also  $F_\varepsilon(r) \leq F_{\varepsilon'}(r + |\varepsilon'| - |\varepsilon|)$ .

Now let  $A$  be a subset of  $[k]^n$ . For  $\varepsilon \in \{-1, 1\}^n$ , define the  $\varepsilon$ -quadrant of  $A$  to be

$$A_\varepsilon = \{x \in Q_\varepsilon : c' + \sum \varepsilon_i(x_i + \frac{1}{2})e_i \in A\},$$

and set

$$s(\varepsilon) = s_A(\varepsilon) = \begin{cases} \max \{ \sum x_i : x \in A_\varepsilon \} & \text{if } A_\varepsilon \neq \emptyset, \\ -1 & \text{if } A_\varepsilon = \emptyset. \end{cases}$$

Thus

$$|A| \leq \sum_\varepsilon \sum_{r=0}^{s(\varepsilon)} F_\varepsilon(r). \quad (3')$$

It is easy to see that if  $A$  is a compressed set and  $\varepsilon, \varepsilon' \in \{-1, 1\}^n$  are adjacent then we have

$$|s_A(\varepsilon') - s_A(\varepsilon)| \leq 1. \quad (4')$$

For any integer  $a$ , we write  $A^{(\geq a)}$  for  $\{\varepsilon \in \{-1, 1\}^n : s_A(\varepsilon) \geq a\}$ .

We are now ready to deal with the odd grid.

**Theorem 11.** *Let  $k$  be odd, and let  $A$  be a subset of  $[k]^n$  of diameter  $d$ . Then*

$$|A| \leq \begin{cases} |B(c, d/2)| & \text{if } d \text{ is even,} \\ |B(c - \frac{1}{2}e_1, d/2)| & \text{if } d \text{ is odd,} \end{cases}$$

where  $c = ((k-1)/2, \dots, (k-1)/2)$ .

**Proof.** Although the case of  $d$  even has already been dealt with by Theorem 6, we give here a self-contained proof for all values of  $d$ .

By Lemma 9, we may assume without loss of generality that  $A$  is compressed. For convenience, set  $B = B(c, d/2)$  if  $d$  is even and  $B = B(c - \frac{1}{2}e_1, d/2)$  if  $d$  is odd. As before, we distinguish two cases, according to whether  $d \geq 2n$  or  $d < 2n$ .

*Case (i):  $d \geq 2n$ .*

Just as before, condition (4') implies that  $s(\varepsilon) = s_A(\varepsilon) \leq d/2$  for all  $\varepsilon$  and  $s(\varepsilon) + s(-\varepsilon) \leq d - n$  for all  $\varepsilon$ .

For simplicity, we shall start by considering the case when  $d \equiv n \pmod{2}$ . Rearranging the sum in (3'), we have

$$|A| \leq \alpha_1 + \sum_{a=(d-n)/2+1}^{\lfloor d/2 \rfloor} \sum_{\varepsilon \in A^{(\geq a)}} (F_\varepsilon(a) - F_{-\varepsilon}(d-n+1-a)),$$

where

$$\alpha_1 = \sum_{r=0}^{(d-n)/2} \sum_{\varepsilon} F_{\varepsilon}(r).$$

A similar relation holds with  $A$  replaced by  $B$ , with the inequality replaced by equality. Now, it follows just as before from Harper's theorem that  $|A^{(>a)}| \leq |B^{(\geq a)}|$  for any  $a$  with  $(d-n)/2 + 1 \leq a \leq d/2$ . Hence, to complete the proof that  $|A| \leq |B|$ , it will suffice to show that for any  $a$  satisfying  $(d-n)/2 + 1 \leq a \leq d/2$  the following two conditions hold:  $\varepsilon \in B^{(\geq a)}$  and  $\varepsilon' \notin B^{(\geq a)}$  imply  $F_{\varepsilon}(a) - F_{-\varepsilon}(d-n+1-a) \geq F_{\varepsilon'}(a) - F_{-\varepsilon'}(d-n+1-a)$ , and  $\varepsilon \in B^{(\geq a)}$  implies  $F_{\varepsilon}(a) - F_{-\varepsilon}(d-n+1-a) \geq 0$ .

Now, since  $F_{\varepsilon}(r)$  is an increasing function of  $|\varepsilon|$ , for any  $r$ , it follows that  $F_{\varepsilon}(a) - F_{-\varepsilon}(d-n+1-a)$  is also an increasing function of  $|\varepsilon|$ . Hence certainly if  $\varepsilon \in B^{(\geq a)}$  and  $\varepsilon' \notin B^{(\geq a)}$  then  $F_{\varepsilon}(a) - F_{-\varepsilon}(d-n+1-a) \geq F_{\varepsilon'}(a) - F_{-\varepsilon'}(d-n+1-a)$ . Also, for any  $\varepsilon, \varepsilon'$  we have  $F_{\varepsilon}(a) \geq F_{\varepsilon'}(d-n+1-a)$  whenever  $|\varepsilon| \geq |\varepsilon'| + 2a - d + n - 1$ , and so in particular  $F_{\varepsilon}(a) \geq F_{-\varepsilon}(d-n+1-a)$  whenever  $\varepsilon \in B^{(\geq a)}$ .

We now turn to the case  $d \not\equiv n \pmod{2}$ . Let

$$S = \{\varepsilon: |\varepsilon| > n/2\} \cup \{\varepsilon: |\varepsilon| = n/2, \varepsilon_1 = -1\}.$$

Thus  $|S \cap \{\varepsilon, -\varepsilon\}| = 1$  for all  $\varepsilon$ . Also, for  $\varepsilon \in S$  we have  $|\varepsilon| \geq |-\varepsilon|$ , which implies  $F_{\varepsilon}((d-n+1)/2) \geq F_{-\varepsilon}((d-n+1)/2)$ . It follows that we may rearrange the sum in (3') to obtain

$$|A| \leq \alpha_2 + \sum_{a=(d-n+1)/2+1}^{\lfloor d/2 \rfloor} (F_{\varepsilon}(a) - F_{-\varepsilon}(d-n+1-a)),$$

where

$$\alpha_2 = \sum_{r=0}^{(d-n)/2} \sum_{\varepsilon} F_{\varepsilon}(r) + \sum_{\varepsilon \in S} F_{\varepsilon}((d-n+1)/2).$$

Note that equality holds in this relation if  $A$  is replaced by  $B$ . It now follows, exactly as in the case  $d \equiv n \pmod{2}$ , that  $|A| \leq |B|$ .

*Case (ii):  $d < 2n$ .*

Rearranging the sum in (3'), we obtain

$$|A| \leq \alpha_1 + \sum_{a=(d-n)/2+1}^{d-n} \sum_{\varepsilon \in A^{(>a)}} (F_{\varepsilon}(a) - F_{-\varepsilon}(d-n+1-a)) + \sum_{\substack{a > d-n \\ a \geq 0}} \sum_{\varepsilon \in A^{(\geq a)}} F_{\varepsilon}(a)$$

if  $d \equiv n \pmod{2}$ , and

$$|A| \leq \alpha_2 + \sum_{a=(d-n+1)/2+1}^{d-n} \sum_{\varepsilon \in A^{(>a)}} (F_{\varepsilon}(a) - F_{-\varepsilon}(d-n+1-a)) + \sum_{\substack{a > d-n \\ a \geq 0}} \sum_{\varepsilon \in A^{(\geq a)}} F_{\varepsilon}(a)$$

if  $d \not\equiv n \pmod{2}$ , where  $\alpha_1$  and  $\alpha_2$  are as in Case (i).

Now, equality holds in this relation if  $A$  is replaced by  $B$ . Also, just as in the proof of Theorem 10, Harper's theorem implies that  $|A^{(\geq a)}| \leq |B^{(\geq a)}|$  for any nonnegative  $a$  with  $a \geq (d-n)/2 + 1$ . Hence, to complete the proof that  $|A| \leq |B|$ , it will suffice to show that for each  $a$  with  $(d-n)/2 + 1 \leq a \leq d-n$  the function  $F_\varepsilon(a) - F_{-\varepsilon}(d-n+1-a)$  is both increasing in  $|\varepsilon|$  and nonnegative on  $B^{(\geq a)}$ , and that for each nonnegative  $a$  with  $a \geq d-n$  the function  $F_\varepsilon(a)$  is increasing in  $|\varepsilon|$ . But these facts are immediate, just as in Case (i).  $\square$

Theorems 8, 10 and 11 imply that, for any  $k$  and  $d$ , there is indeed a ball among the maximum-sized subsets of  $[k]^n$  of diameter  $d$ , as conjectured by Kleitman and Fellows.

**Corollary 12.** *Let  $A$  be a subset of  $[k]^n$  of diameter  $d$ , where  $d \in \mathbb{Z}_+$ . Then there exists  $c \in \mathbb{R}^n$  such that  $|A| \leq |B(c, d/2)|$ .*

### 3. Subsets of $\mathbb{Z}_k^n$

Given a set  $A$  of vertices of a graph  $G$  and a natural number  $t$ , define the  $t$ -boundary of  $A$  as

$$A_{(t)} = \{x \in V(G) : d(x, A) \leq t\},$$

where as usual  $d(x, A)$  denotes the graph distance from  $x$  to  $A$ . We often write  $\partial A$  for  $A_{(1)}$ , and call it the *boundary* of  $A$ .

An inequality of the form

$$|A_{(t)}| \geq g(a, t) \quad \text{whenever } A \subset V(G) \text{ and } |A| = a$$

is called an *isoperimetric inequality* on  $G$ . Our approach to the diameter problem in  $\mathbb{Z}_k^n$  will be based on some isoperimetric inequalities in  $\mathbb{Z}_k^n$ . Isoperimetric inequalities in  $\mathbb{Z}_k^n$  were first proved in [3], in answer to a question of Wang and Wang [8]. (The results proved in [8] concern some rather different isoperimetric problems.) The inequalities we obtain here slightly extend those in [3].

A *fractional system*, or simply a *system*, on  $\mathbb{Z}_k^n$  is a function from  $\mathbb{Z}_k^n$  to the closed interval  $[0, 1]$ . The *boundary* of a system  $f$  is the system  $\partial f$  given by

$$\partial f(x) = \begin{cases} 1 & \text{if } f(x) > 0, \\ \max\{f(y) : d(y, x) = 1\} & \text{if } f(x) = 0. \end{cases}$$

Thus if  $f(\mathbb{Z}_k^n) \subset \{0, 1\}$  and  $f$  is identified with  $A = f^{-1}(1)$  then  $\partial f$  is identified with the usual boundary of  $A$ .

A system  $f$  which is of the form

$$f(x) = \begin{cases} 1 & \text{if } d(x, 0) < r, \\ \alpha & \text{if } d(x, 0) = r, \\ 0 & \text{if } d(x, 0) > r, \end{cases}$$

for some  $0 \leq r \leq nk/2$  and  $\alpha \in [0, 1]$  is called a *fractional ball*. The *radius* of  $f$  is  $r + \alpha - 1$ . For  $0 \leq v \leq k^n$  we write  $b^v$  for the (unique) fractional ball of weight  $v$ .

Our aim is to show that, for *even* values of  $k$ , fractional balls in  $\mathbb{Z}_k^n$  have the smallest boundaries, in other words that the weight of the boundary of a system of weight  $v$  is at least  $w(\partial b^v)$ . This was first proved in [3]; here we give different proof. We include a proof not only for the sake of completeness but also because we shall later require a refined version of this result; the proof we give here may be readily modified to give that refinement.

We start with the simple case  $n=1$ ; although in this case the assertion is easily checked, we state it as a lemma in order to emphasize it. Furthermore, we give a proof, since the corresponding assertion is false for odd values of  $k$ .

**Lemma 13.** *Let  $k \geq 2$  be even. Then in  $\mathbb{Z}_k$  the weight of the boundary of a system  $f$  of weight  $v$  is at least the weight of the boundary of the fractional ball  $b^v$  of weight  $v$ .*

**Proof.** If  $|\text{supp } f| = k$  then  $\partial f$  is identically 1, so that certainly  $w(\partial b^v) \leq w(\partial f)$ .

If  $|\text{supp } f| = k-1$ , say  $|\text{supp } f| = \mathbb{Z}_k - \{k/2\}$ , then  $\partial f(x) = 1$  for  $x \neq k/2$ , and  $\partial f(k/2) = \max(f(k/2-1), f(k/2+1))$ . Since  $b^v(k/2-1) = b^v(k/2+1)$  and  $b^v(k/2-1) + b^v(k/2+1) \leq f(k/2-1) + f(k/2+1)$ , it follows that  $\partial b^v \leq \partial f$ , so that  $w(\partial b^v) \leq w(\partial f)$ .

If  $|\text{supp } f| \leq 1$ , then  $w(\partial b^v) = w(\partial f)$ .

Finally, if  $2 \leq |\text{supp } f| \leq k-2$  then it is easy to see that  $w(\partial f) \geq w(f) + 2$ . However, for all fractional balls  $b$  we have  $w(\partial b) \leq w(b) + 2$ .  $\square$

We remark that if  $k \geq 3$  is odd then, in  $\mathbb{Z}_k$ , a fractional ball of weight  $k-2$  has greater boundary than the complement of a fractional ball of weight 2.

Let  $f$  be a system on  $\mathbb{Z}_k^n$ . For  $I \subset \{1, \dots, n\}$  and  $x \in \mathbb{Z}_k^I$ , the  $I$ -section of  $f$  at  $x$  is the system  $f_{I|x}$  on  $\mathbb{Z}^I$  given by

$$f_{I|x}(y) = f(x+y), \quad y \in \mathbb{Z}^I.$$

We wish to define a compression operator  $C_i$  which acts on a system  $f$  by ‘compressing’ each  $i$ -section of  $f$  into a (1-dimensional) fractional ball of the same weight. Thus, for a system  $f$  on  $\mathbb{Z}_k^n$ , we define the system  $C_i(f)$ , the  $i$ -compression of  $f$ , by giving its  $i$ -sections:

$$C_i(f)_{i|x} = b^{w(f_{i|x})}, \quad x \in \mathbb{Z}_k^{\bar{i}}.$$

Thus  $w(C_i(f)) = w(f)$ .

The reason for introducing  $i$ -compressions is the following lemma.

**Lemma 14.** *Let  $k \geq 2$  be even. Let  $f$  be a system on  $\mathbb{Z}_k^n$ , and let  $1 \leq i \leq n$ . Then  $w(\partial C_i(f)) \leq w(\partial f)$ .*

**Proof.** Write  $g$  for  $C_i(f)$ . It is sufficient to show that  $w((\partial g)_{i|x}) \leq w((\partial f)_{i|x})$  for all  $x \in \mathbb{Z}_k^{\bar{i}}$ . Fix then an arbitrary  $x \in \mathbb{Z}_k^{\bar{i}}$ . We have

$$(\partial f)_{i|x} = \partial(f_{i|x}) \vee \bigvee \{f_{i|y} : y \in \mathbb{Z}_k^{\bar{i}}, d(y, x) = 1\}$$

where as defined earlier  $\vee$  denotes pointwise maximum, and a similar relation holds with  $g$  in place of  $f$ . However, the system  $\partial(g_{i|x})$  and the systems  $g_{i|y}$  for  $y \in \mathbb{Z}_k^i$ ,  $d(y, x) = 1$  are nested, since they are all fractional balls. Moreover,  $w(g_{i|y}) = w(f_{i|y})$  for all  $y$ , while Lemma 13 tells us the  $w(\partial(g_{i|x})) \leq w(\partial(f_{i|x}))$ . It follows that  $w((\partial g)_{i|x}) \leq w((\partial f)_{i|x})$ , as required.  $\square$

We now wish to define a symmetrisation operator  $S_i$  which acts on a system  $f$  by replacing each  $i$ -section of  $f$  with an  $((n-1)$ -dimensional) fractional ball of the same weight. Thus, for a system  $f$  on  $\mathbb{Z}_k^n$ , we define the system  $S_i(f)$ , the  $i$ -symmetrisation of  $f$ , by giving its  $i$ -sections:

$$S_i(f)_{i|x} = b^{w(f_{i|x})}, \quad x \in \mathbb{Z}_k.$$

Thus  $w(S_i(f)) = w(f)$ .

We have the following key lemma.

**Lemma 15.** *Let  $k \geq 2$  be even, and let  $n \geq 2$ . Suppose that for every system  $f$  on  $\mathbb{Z}_k^{n-1}$  we have  $w(\partial f) \geq w(\partial b^{w(f)})$ . Then for a system  $f$  on  $\mathbb{Z}_k^n$  and  $1 \leq i \leq n$  we have  $w(\partial S_i(f)) \leq w(\partial f)$ .*

**Proof.** Write  $g$  for  $S_i(f)$ . It is sufficient to show that  $w((\partial g)_{i|x}) \leq w((\partial f)_{i|x})$  for all  $x \in \mathbb{Z}_k$ . Fix then an arbitrary  $x \in \mathbb{Z}_k$ . We have

$$(\partial f)_{i|x} = \partial(f_{i|x}) \vee \bigvee \{f_{i|y} : y \in \mathbb{Z}_k, d(y, x) = 1\},$$

and a similar relation holds with  $g$  in place of  $f$ . However, the system  $\partial(g_{i|x})$  and the systems  $g_{i|y}$  for  $y \in \mathbb{Z}_k$ ,  $d(y, x) = 1$  are nested, since they are all fractional balls. Moreover,  $w(g_{i|y}) = w(f_{i|y})$  for all  $y$ , and by assumption we have  $w(\partial(g_{i|x})) \leq w(\partial(f_{i|x}))$ . It follows that  $w((\partial g)_{i|x}) \leq w((\partial f)_{i|x})$ , as required.  $\square$

We are now ready to prove the isoperimetric inequality in the even torus. As stated above, this was first proved in [3].

**Theorem 16.** *Let  $k \geq 2$  be even, and let  $f$  be a system on  $\mathbb{Z}_k^n$  of weight  $v$ . Then  $w(\partial f) \geq w(\partial b^v)$ .*

**Proof.** We proceed by induction on  $n$ . For  $n=1$  the assertion was proved in Lemma 13, so we turn to the induction step.

For any system  $g$  on  $\mathbb{Z}_k^n$  and  $x \in \mathbb{Z}_k$ , write  $g_x$  for  $g_{i|x}$ . Let  $f' = S_1(C_1(f))$ . Then  $w(f') = w(f)$ , and from Lemma 14 and Lemma 15 we have  $w(\partial f') \leq w(\partial f)$ . Moreover,  $f'_x$  is a fractional ball for each  $x \in \mathbb{Z}_k$ , with  $f'_x = f'_{-x}$  for all  $x$  and  $f'_x \geq f'_y$  whenever  $d(x, 0) < d(y, 0)$ .



Define

$$G = \{g \in [0, 1]^{\mathbb{Z}_k^*} : w(g) = w(f), w(\partial g) \leq w(\partial f), g_x \text{ a fractional ball for all } x, \\ \text{with } g_x = g_{-x}, \text{ and } g_x \geq g_y \text{ when } d(x, 0) < d(y, 0)\}.$$

We shall show that there is a fractional ball in  $G$ . Note first that  $G$  is a compact subset of the product space  $[0, 1]^{\mathbb{Z}_k^*}$ , and is non-empty as  $f' \in G$ .

For  $0 \leq r \leq nk/2$ , write  $w^{(r)}(g)$  for  $\sum_{d(x, 0)=r} g(x)$ , and let  $H$  be the subset of  $G$  obtained by maximising successively  $w^{(0)}(g), w^{(1)}(g), \dots, w^{(nk/2)}(g)$  on  $G$ . Since  $G$  is compact,  $H$  is non-empty. Choose  $g \in H$ . We claim that in fact  $g$  is a fractional ball.

Let the radius of the fractional ball  $g_x$  be  $r_x$ . For each  $0 \leq x < k/2$ , we shall show that  $r_{x+1} \geq r_x - 1$ , and that  $r_{x+1} = r_x - 1$  unless either  $w(g_{x+1}) = 0$  or  $w(g_x) = k^{n-1}$ . From this it follows immediately that  $g$  is a fractional ball.

Fix then an arbitrary  $0 \leq x < k/2$ . For simplicity, we shall consider first the case  $0 < x < k/2 - 1$ . Suppose, for a contradiction, that  $r_{x+1} < r_x - 1$ . Then, for  $\varepsilon > 0$  sufficiently small, we can define a system  $g'$  on  $\mathbb{Z}_k^n$  by

$$g'_y = \begin{cases} g_y & \text{if } y \neq \pm x, \pm(x+1), \\ b^{w(g_x) - \varepsilon} & \text{if } y = \pm x, \\ b^{w(g_{x+1}) + \varepsilon} & \text{if } y = \pm(x+1). \end{cases}$$

Then  $w(g') = w(g)$ . Now,  $(\partial g')_y = (\partial g)_y$  if  $y \neq \pm x, \pm(x+1), \pm(x+2)$ , and  $(\partial g')_x \leq (\partial g)_x$ . Also, if  $\varepsilon$  is sufficiently small then  $w((\partial g')_{x+1}) = w((\partial g)_{x+1}) - \varepsilon$ , and  $w((\partial g')_{x+2}) \leq w((\partial g)_{x+2}) + \varepsilon$ . It follows that  $w(\partial g') \leq w(\partial g)$ , whence  $g' \in G$ . However, this contradicts  $g \in H$ .

If  $x = 0$  or  $k/2 - 1$ , the argument is similar. Thus  $r_{x+1} \geq r_x - 1$  for all  $0 \leq x < k/2$ .

We now show that  $r_{x+1} = r_x - 1$  unless either  $w(g_{x+1}) = 0$  or  $w(g_x) = k^{n-1}$ . As above, we consider first the case  $0 < x < k/2 - 1$ . Suppose, for a contradiction, that  $r_{x+1} > r_x - 1$ , with  $w(g_{x+1}) > 0$  and  $w(g_x) < k^{n-1}$ . Then, for  $\varepsilon > 0$  sufficiently small, we can define a system  $g'$  on  $\mathbb{Z}_k^n$  by

$$g'_y = \begin{cases} g_y & \text{if } y \neq \pm x, \pm(x+1), \\ b^{w(g_x) - \varepsilon} & \text{if } y = \pm x, \\ b^{w(g_{x+1}) + \varepsilon} & \text{if } y = \pm(x+1). \end{cases}$$

Then  $w(g') = w(g)$ . We have  $(\partial g')_y = (\partial g)_y$  if  $y \neq \pm x, \pm(x+1), \pm(x+2)$ , and  $(\partial g')_{x+2} \leq (\partial g)_{x+2}$ . Also, if  $\varepsilon$  is sufficiently small then  $(\partial g')_{x+1} = \partial(g'_{x+1})$ , and  $(\partial g')_x \leq \partial(g'_x)$ .

Now, the function

$$s \mapsto |\{z \in \mathbb{Z}_k^z : d(z, 0) = s\}|$$

is log concave, because the function

$$s \mapsto \left| \left\{ z \in \left[ \frac{k}{2} \right]^n : \sum z_i = s \right\} \right|$$

is log concave (see e.g. [1, Ch. 4]). From this, it is easy to see that

$$w((\partial g')_x) - w((\partial g)_x) \leq w((\partial g)_{x+1}) - w((\partial g')_{x+1}).$$

It follows that  $w(\partial g') \leq w(g)$ , and consequently  $g' \in G$ . However, this contradicts  $g \in H$ .

If  $x=0$  or  $k/2-1$ , we obtain a contradiction by a similar argument.  $\square$

In particular, we have the following result for (non-fractional) subsets of the torus.

**Corollary 17.** *Let  $k$  be even and let  $A$  be a subset of  $\mathbb{Z}_k^n$  with  $|A| \geq |B(0, r)|$ , where  $r \in \mathbb{Z}_+$ . Then  $|\partial A| \geq |B(0, r+1)|$ .*

From this we deduce immediately the corresponding result about  $t$ -boundaries.

**Corollary 18.** *Let  $k$  be even and let  $A$  be a subset of  $\mathbb{Z}_k^n$  with  $|A| \geq |B(0, r)|$ , where  $r \in \mathbb{Z}_+$ . Then for any  $t=0, 1, \dots$  we have  $|A_{(t)}| \geq |B(0, r+t)|$ .*

As we shall now see, this solution to the isoperimetric problem in  $\mathbb{Z}_k^n$  implies a solution to the diameter problem in  $\mathbb{Z}_k^n$ .

**Theorem 19.** *Let  $k$  be even, and let  $A$  be a subset of  $\mathbb{Z}_k^n$  of diameter  $d$ , where  $d$  is even. Then*

$$|A| \leq \begin{cases} |B(0, d/2)| & \text{if } d < nk/2, \\ |\mathbb{Z}_k^n| & \text{if } d = nk/2. \end{cases}$$

**Proof.** We may and shall assume  $d < nk/2$ . For  $x \in \mathbb{Z}_k^n$ , write  $\bar{x}$  for  $x + \sum (k/2)e_i$ , and let  $\bar{A} = \{\bar{x} : x \in A\}$ .

For  $x, y \in A$ , we have  $d(x, y) \leq d$ , so that  $d(x, \bar{y}) \geq nk/2 - d$ . It follows that  $d(A, \bar{A}) \geq nk/2 - d$ . Thus

$$A_{(nk/2-d-1)} \subset \mathbb{Z}_k^n - \bar{A}. \quad (5)$$

Suppose for a contradiction that  $|A| > |B(0, d/2)|$ . Then, from Corollary 18, we have

$$\begin{aligned} |A_{(nk/2-d-1)}| &\geq |B(0, nk/2-d/2-1)| \\ &= |\mathbb{Z}_k^n - B(\bar{0}, d/2)| \\ &> |\mathbb{Z}_k^n - \bar{A}|, \end{aligned}$$

which contradicts (5).  $\square$

It is interesting to note that, in the case  $k=2$ , Corollary 18 is precisely Harper's vertex-isoperimetric inequality in the discrete cube [5], and Theorem 19 is Kleitman's solution to the diameter problem in the cube [6]. Thus, as shown by the proof of Theorem 19, Harper's theorem immediately implies Kleitman's theorem.

We now turn to the case of odd diameter. For  $x, y \in (\mathbb{R}/k\mathbb{Z})^n$ , we write  $d(x, y)$  for  $\min\{\sum |x_i - y_i + a_i| : a_i \in k\mathbb{Z} \text{ for all } i\}$ . A system  $f$  on the one-dimensional torus  $\mathbb{Z}_k$  which is of the form

$$f(x) = \begin{cases} 1 & \text{if } d(x, 1/2) < r + 1/2, \\ \alpha & \text{if } d(x, 1/2) = r + 1/2, \\ 0 & \text{if } d(x, 1/2) > r + 1/2, \end{cases}$$

for some  $0 \leq r \leq k/2$  and  $\alpha \in [0, 1]$  is called a *shifted fractional ball*. For  $0 \leq v \leq k$  we write  $s^v$  for the unique shifted fractional ball of weight  $v$ .

We wish to define a compression operator  $D_i$  which acts on a system  $f$  by replacing each  $i$ -section of  $f$  with a shifted fractional ball of the same weight. Thus, for a system  $f$  on  $\mathbb{Z}_k^n$ , the system  $D_i(f)$ , the *shifted  $i$ -compression* of  $f$ , is defined by:

$$D(f)_{i|x} = s^{w(f_{i|x})}, \quad x \in \mathbb{Z}_k^i.$$

Thus  $w(D_i(f)) = w(f)$ .

Applying an operator  $D_i$  can certainly increase the boundary of a system. However, it is our good fortune that this does not happen when  $D_i$  is applied to a (non-fractional) subset of  $\mathbb{Z}_k^n$ .

**Lemma 20.** *Let  $k \geq 2$  be even. Let  $f$  be a system on  $\mathbb{Z}_k^n$  with  $f(\mathbb{Z}_k^n) \subset \{0, 1\}$ , and let  $1 \leq i \leq n$ . Then  $w(\partial D_i(f)) \leq w(\partial f)$ .*

**Proof.** It is easy to check that if  $g$  is a system on the one-dimensional torus  $\mathbb{Z}_k$  with  $g(\mathbb{Z}_k) \subset \{0, 1\}$  then  $w(\partial s^{w(g)}) \leq w(\partial g)$ . The result now follows from this remark exactly as Lemma 14 followed from Lemma 13.  $\square$

For  $c \in (\mathbb{R}/k\mathbb{Z})^n$  and  $r > 0$ , let  $B(c, r) = \{x \in \mathbb{Z}_k^n : d(x, c) \leq r\}$ . The following result should be compared with Corollary 17.

**Theorem 21.** *Let  $k$  be even and let  $A$  be a subset of  $\mathbb{Z}_k^n$  with  $|A| \geq |B(\frac{1}{2}e_1, r + \frac{1}{2})|$ , where  $r \in \mathbb{Z}_+$ . Then  $|\partial A| \geq |B(\frac{1}{2}e_1, r + 1 + \frac{1}{2})|$ .*

**Proof.** As before, write  $g_x$  for  $g_{i|x}$ . Let  $f' = S_1(D_1(A))$ . Then  $w(f') = w(A)$ , and from Lemma 20 and Lemma 15 we have  $w(\partial f') \leq w(\partial A)$ . Moreover,  $f'_x$  is a fractional ball for each  $x \in \mathbb{Z}_k$ , with  $f'_x = f'_{1-x}$  for all  $x$  and  $f'_x \geq f'_y$  whenever  $d(x, \frac{1}{2}) < d(y, \frac{1}{2})$ .

Define

$$G = \{g \in [0, 1]^{\mathbb{Z}_k^i} : w(g) = w(A), w(\partial g) \leq w(\partial A), g_x \text{ a fractional ball for all } x, \\ \text{with } g_x = g_{1-x}, \text{ and } g_x \geq g_y \text{ when } d(x, \frac{1}{2}) < d(y, \frac{1}{2})\}.$$

The  $G$  is a non-empty subset of  $[0, 1]^{\mathbb{Z}_k^i}$ .

For  $0 \leq s \leq nk/2 - 1$ , write  $w^{(r)}(g)$  for  $\sum_{d(x, e_1/2) = s + 1/2} g(x)$ , and let  $H$  be the subset of  $G$  obtained by maximising successively  $w^{(0)}(g), w^{(1)}(g), \dots, w^{(nk/2-1)}(g)$  on  $G$ . Since  $G$  is compact,  $H$  is non-empty. Choose  $g \in H$ . Then, exactly as in the proof of

Theorem 16, it follows that  $g$  is of the form

$$g(x) = \begin{cases} 1 & \text{if } d(x, e_1/2) < s + 1/2, \\ \alpha & \text{if } d(x, e_1/2) = s + 1/2, \\ 0 & \text{if } d(x, e_1/2) > s + 1/2, \end{cases}$$

for some  $0 \leq s \leq k/2$  and  $\alpha \in [0, 1]$ .  $\square$

From this we deduce immediately the corresponding result about  $t$ -boundaries.

**Corollary 22.** *Let  $k$  be even and let  $A$  be a subset of  $\mathbb{Z}_k^n$  with  $|A| \geq |B(\frac{1}{2}e_1, r + \frac{1}{2})|$ , where  $r \in \mathbb{Z}_+$ . Then for any  $t = 0, 1, \dots$  we have  $|A_{(t)}| \geq |B(\frac{1}{2}e_1, r + t + \frac{1}{2})|$ .*

We are now ready to find the maximal size of a set of given odd diameter in the even torus.

**Theorem 23.** *Let  $k$  be even, and let  $A$  be a subset of  $\mathbb{Z}_k^n$  of diameter  $d$ , where  $d$  is odd. Then*

$$|A| \leq \begin{cases} |B(\frac{1}{2}e_1, d/2)| & \text{if } d < nk/2, \\ |\mathbb{Z}_k^n| & \text{if } d = nk/2. \end{cases}$$

**Proof.** Mirror the proof of Theorem 19, the corresponding result for even diameter.  $\square$

In conclusion, let us turn to the case when  $k$  is odd. As the best isoperimetric inequality in the odd torus is still not known, we cannot apply the above methods to tackle the diameter problem in the odd torus. It was conjectured in [3] that balls centred at 0, or their complements, have smallest boundary — in other words, that if  $A \subset \mathbb{Z}_k^n$  satisfies  $|A| \geq |B(0, r)|$  and  $|A| \geq k^n - |B(0, s)|$  for some integers  $r$  and  $s$ , then its boundary  $\partial A$  satisfies either  $|\partial A| \geq |B(0, r+1)|$  or  $|\partial A| \geq k^n - |B(0, s-1)|$ . However, even if this conjecture were proved, we would still be unable to apply the methods above, as there is a more serious problem: a point  $x$  in the odd torus  $\mathbb{Z}_k^n$  does not have an ‘opposite’ point  $\bar{x}$ .

In fact, for  $k$  odd it is not always true that there is a ball among the maximum-sized subsets of  $\mathbb{Z}_k^n$  of diameter  $d$ . This is not even the case when  $d$  is even. For example, the set  $\{(x, y) : x = 0 \text{ or } 1\} \subset \mathbb{Z}_7^2$  has size 14 and diameter 4, whereas every ball of radius 2 in  $\mathbb{Z}_7^2$  has size less than 14: the balls  $B(0, 2)$  and  $B((\frac{1}{2}, \frac{1}{2}), 2)$  have sizes 13 and 12 respectively. For odd values of  $k$ , we do not know the maximum size of a subset of  $\mathbb{Z}_k^n$  of diameter  $d$ .

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